Welcome to Section 6.5 - Convolutions

Definition of a Convolution

If f(t) and g(t) are functions with Laplace transforms F(s) and G(s), then the (f * g)(t) is defined as the **convolution** of f(t) & g(t) where

$$(f * g)(t) = \int_0^t f(t-u)\, g(u)\, du\ .$$

By using the integral definition of the Laplace transform of f(t) and g(t), and applying a variable substitution (as shown in the text), it is shown that

$$F(s)G(s) = \mathcal{L}[f * g]$$

AND

$$\mathcal{L}^{-1}[F(s)G(s)] = f * g\ .$$

In other words, the inverse Laplace transform of a product of Laplace functions of s will equal the convolution of f(t) and g(t).

Example: If f(t) = e^t , and g(t) = 1, compute (f * g)(t).

$$(f * g)(t) = \int_0^t e^{t-u} \cdot 1\, du$$

$$= -e^{t-u}\Big]_0^t = -\left[e^{t-t} - e^{t-0}\right]$$

$$= -1 + e^t$$

Note: In this case, (g*t)(t) also results in -1 + e^t. It may also be shown that in general, (f*g)(t) = (g*t)(t).

Example: Use the convolution Theorem to find $\mathcal{L}^{-1}\left[\frac{1}{s(s+1)}\right]$.

$$\mathcal{L}^{-1}\left[\frac{1}{s(s+1)}\right] = \mathcal{L}^{-1}\left[\frac{1}{s} \cdot \frac{1}{s+1}\right]$$

where F(s) = 1/s and G(s) = 1/(s+1). The corresponding functions of t are f(t) = 1, and g(t) = e^{-t}.

Continued. . .

$$\mathcal{L}^{-1}[F(s)G(s)] = f*g$$

$$= \int_0^t 1\cdot e^{-u}\,du = -e^{-u}\Big]_0^t = -\left[e^{-t}-e^{0}\right]$$

$$= 1-e^{-t}$$

Unfortunately, most integrals representing the convolution of 2 functions are not this easy to evaluate. The convolution has other applications.

Solving a Differential Equation with Convolution

Suppose we wish to solve the second-order differential equation

$$y'' + y' - 2y = f(t) \qquad \text{WITH } y(0)=y'(0)=0$$

where f(t) is some forcing function. Take Laplace transforms of both sides and solve for L[y] in terms of L[f(t)] and s. Continued. . .

Taking Laplace transforms and substituting in y(0)=0 & y'(0)=0 results in

$$s^2\mathcal{L}[y] + s\mathcal{L}[y] - 2\mathcal{L}[y] = \mathcal{L}[f(t)].$$

Solve for L[y].

$$\mathcal{L}[y] = \mathcal{L}[y]\cdot\frac{1}{s^2+s-2}$$

Take the inverse Laplace transform of both sides.

$$y = \mathcal{L}^{-1}\left[\mathcal{L}[f(t)]\cdot\frac{1}{s^2+s-2}\right]$$

Applying the definition of the convolution results in

$$y = f(t) * \mathcal{L}^{-1}\left[\frac{1}{s^2+s-2}\right].$$

We may use partial fractions to rewrite the fraction.

$$y = f(t) * \mathcal{L}^{-1}\left[\frac{1/3}{s-2} - \frac{1/3}{s+1}\right]$$

Take the inverse Laplace of the fractions.

$$y = f(t) * \left[\tfrac{1}{3}e^{2t} - \tfrac{1}{3}e^{-t}\right] \quad \text{CONTINUED}$$

Applying the integral definition of the convolution results in

$$y = \int_0^t \left\{ \frac{1}{3} e^{2(t-u)} - \frac{1}{3} e^{-(t-u)} \right\} f(u)\, du .$$

Direc delta function at t=0

Notice that if you solved y" + y' - 2y = δ_0 , with y(0)=y'(0)=0, you would take Laplace transforms of both sides to get . . .

$$s^2 + s\mathcal{L}[y] - 2\mathcal{L}[y] = 1$$

Note that the Laplace of this Direc delta function is $e^0 = 1$.

Solving for L[y] results in

$$\mathcal{L}[y] = \frac{1}{s^2+s-2} = \frac{\frac{1}{3}}{s-2} - \frac{\frac{1}{3}}{s+1} .$$

Applying the inverse Laplace transform results in

$$g(t) = \tfrac{1}{3} e^{2t} - \tfrac{1}{3} e^{-t} .$$

CONTINUED

We can generalize this to obtain the following formula.

$$y = \int_0^t g(t-u)\, f(u)\, du$$

is the solution to $y'' + py' + qy = f(t)$

where y"(0)=y(0)=0 AND $g(t)$ IS A SOLUTION TO $y'' + py' + qy = \delta_0$

and f(t) is any forcing function.

Thus, once we know how a second order system behaves with a Direc delta force at time t=0, we have an integral solution for the behavior of the system for ANY other forcing function f(t).

Do the homework for 6.5 . Read the text!