Welcome to Section 3.5 - Repeated and Zero Eigenvalues

Since the equation used to find the eigenvalues (called the characteristic equation) is a quadratic equation, the coefficients of a system may be such that there is a repeated solution or a zero solution to this equation.

Zero Eigenvalues

In the system dx/dt = 4x - 2y, dy/dt = -2x + y, the characteristic equation and eigenvalues are:

$$(4-\lambda)(1-\lambda)-4=0$$
$$4-5\lambda+\lambda^2-4=0$$
$$\lambda^2-5\lambda=0 \longrightarrow \lambda_1=5,\ \lambda_2=0$$

We may use these eigenvalues to find 2 eigenvectors in the same way as before. We may then form a general solution in the same way as before.

Find the eigenvectors and the general solution for this system.

FOR $\lambda_1=5$

$$4x^*-2y^*=5x^*$$
$$-2x^*+y^*=5y^*$$

$\longrightarrow$ ANY VECTOR WHERE $x^*=-2y^*$ WILL BE AN EIGENVECTOR

$\begin{bmatrix}-2\\1\end{bmatrix}$ IS A GOOD CHOICE

FOR $\lambda_2=0$

$$4x^*-2y^*=0$$
$$-2x^*+y^*=0$$

$\longrightarrow$ ANY VECTOR WHERE $y^*=2x^*$ IS AN EIGENVECTOR.

$\begin{bmatrix}1\\2\end{bmatrix}$ IS A GOOD CHOICE

THE GENERAL SOLUTION IS

$$Y(t)=k_1e^{5t}\begin{bmatrix}-2\\1\end{bmatrix}+k_2e^{0t}\begin{bmatrix}1\\2\end{bmatrix}$$

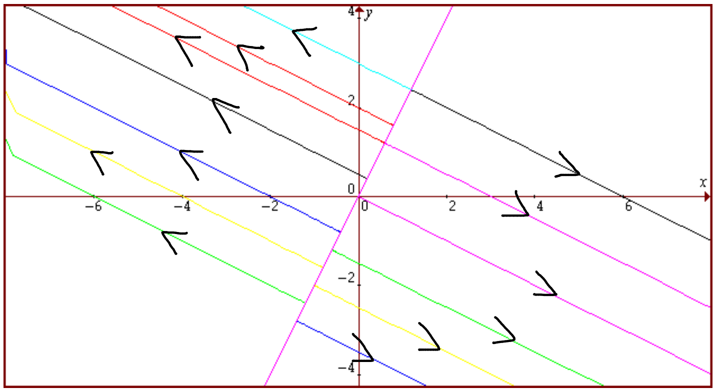
(e^{0t} =1)

OR $Y(t)=k_1e^{5t}\begin{bmatrix}-2\\1\end{bmatrix}+k_2\begin{bmatrix}1\\2\end{bmatrix}$

THIS RESULTS IN $x=-2k_1e^{5t}+k_2$

$y=k_1e^{5t}+2k_2$

The phase portrait for this solution is given below. What we are seeing here are straight-line solutions that are shifted up and down according to what the initial conditions are. We also see a line (y=2x) of equilibrium points. By letting the system = (0,0), any point such that y=2x results in dx/dt =0 & dy/dt =0. Also, if you input any initial conditions into the system, the field vector given by dx/dt and dy/dt indicates that the straight-line solutions move away from the line y=2x.



In general, when solving a linear system with a zero eigenvalue:

1. Find the eigenvalues and associated eigenvectors as before.

2. The general solution is $Y(t) = k_1 e^{\lambda_1 t} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + k_2 \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$

WHERE $\lambda_1 =$ NONZERO EIGENVALUE
AND $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$ IS ASSOCIATED EIGENVECTOR.

$\lambda_2 = 0$ (THUS, $e^{0t} = 1$)
AND $\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ IS THE ASSOCIATED EIGENVECTOR

3. Equilibrium points will lie on a line calculated by letting the system = (0,0). Straight line solutions will move away from this equilibrium line (positive eigenvalue) or toward this equilibrium line (negative eigenvalue). The slope of these straight-line solutions is determined by the eigenvector associated with the non-zero eigenvalue.

Systems With Repeated Eigenvalues

In a system with a repeated eigenvalue, we may easily obtain a single solution to the system:

$$Y(t) = e^{\lambda t}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \quad \text{WHERE } \lambda = \text{THE EIGENVALUE}$$

$$\text{AND } \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \text{ IS THE ASSOCIATED EIGENVECTOR}$$

Note that this is a straight-line solution.

It is shown in the text (quite well) that

the general solution to AY = dY/dt is. . .

BIG RESULT

$$Y(t) = k_1 e^{\lambda t}\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + k_2 e^{\lambda t}\left(t\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right)$$

WHERE $\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}$ IS A VECTOR SUCH THAT $[A]\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} - \lambda\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}$.

NOTE THAT $[A]$ IS THE COEFFICIENT MATRIX.

Example: Find the general solution to the system dx/dt = 2x + y, dy/dt = 2y.

The repeated eigenvalue is 2. The equations used to find the eigenvector (x1,y1) are 2x1 + y1 = 2x1, 2y1 = 2y1. This means that y1=0 and x1 is any real number. A good choice for this eigenvector is $\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Now find a second vector (x2,y2) such that

$$\begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} - 2\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 2x_2 + y_2 \\ 2y_2 \end{bmatrix} - \begin{bmatrix} 2x_2 \\ 2y_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} y_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow \begin{array}{l} y_2 = 1 \\ x_2 = \text{ANY REAL NUMBER.} \\ \text{LET } x_2 = 0 \end{array}$$

$$\begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

The general solution is. . .

$$Y(t) = k_1 e^{2t}\begin{bmatrix}1\\0\end{bmatrix} + k_2 e^{2t}\left(t\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$Y(t) = k_1 e^{2t}\begin{bmatrix}1\\0\end{bmatrix} + k_2 e^{2t}\left(\begin{bmatrix}t\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}\right)$$

$$Y(t) = k_1 e^{2t}\begin{bmatrix}1\\0\end{bmatrix} + k_2 e^{2t}\begin{bmatrix}t\\1\end{bmatrix}$$

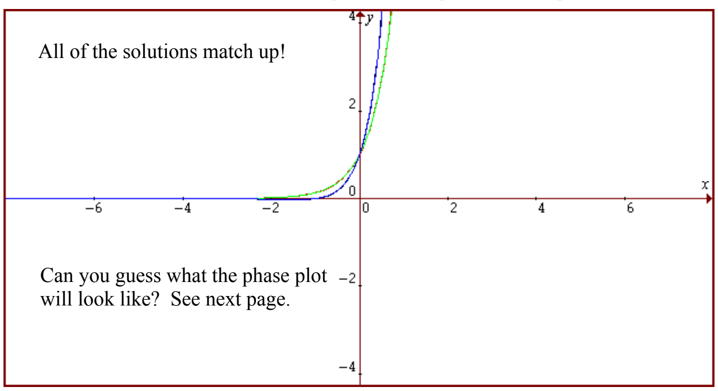
$$x = k_1 e^{2t} + t \cdot k_2 e^{2t}$$
$$y = k_2 e^{2t}$$

Now find the solution with initial conditions {t=0,x=1,y=1}. Check against the solution on Graphmatica.

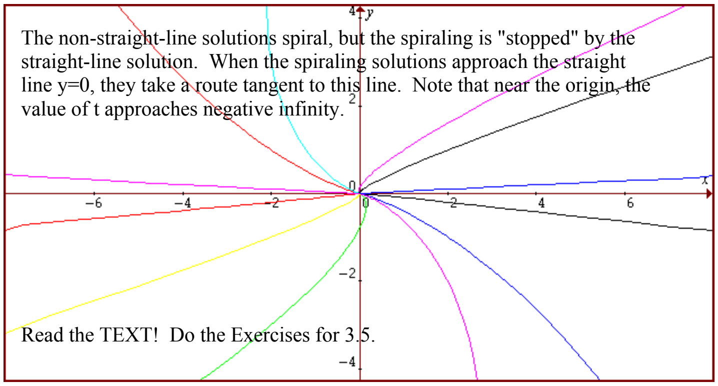
The solutions for k1 & k2 are k1 = 1, k2 = 1. The solution with these initial conditions is

$$x = e^{2t} + te^{2t}$$
$$y = e^{2t}$$

Enter dx=2x+y; dy=2y {0,1,1} , y=exp(2x) + x*exp(2x), & y=exp(2x)



In the phase portrait below, there are straight-line solutions that emanate out from the origin (the origin is a source because of the positive eigenvalue). The other solutions spiral out. The straight line solutions are given by $Y(t) = k_1 e^{2t} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.



The non-straight-line solutions spiral, but the spiraling is "stopped" by the straight-line solution. When the spiraling solutions approach the straight line y=0, they take a route tangent to this line. Note that near the origin, the value of t approaches negative infinity.

Read the TEXT! Do the Exercises for 3.5.