SECTION 1.2

Separation of Variables

An equation of the form dy/dt = f(t,y) may possibly be solved by a technique called "separation of variables" if f(t,y) may be written as the product of a function of t only and a function of y only, that is f(t,y) = g(t)*h(y), separation of variables and integration (if possible) will result in a general solution.

EXAMPLE: SOLVE $\frac{dy}{dt} = yt^2$

$\frac{dy}{y} = t^2\,dt$ (SEPARATE VARIABLES)

NOW, INTEGRATE BOTH SIDES

$$\int \frac{dy}{y} = \int t^2\,dt$$

$$\ln y + C_1 = \frac{t^3}{3} + C_2$$

OR $\ln y = \frac{t^3}{3} + C_3$ WHERE $C_3 = C_2 - C_1$

$e^{\frac{t^3}{3} + C_3} = y$ EXPONENTIAL FORM

$e^{t^3/3} \cdot e^{C_3} = y$

$Ce^{t^3/3} = y$ WHERE $C = e^{C_3}$

Thus, in the last problem, the general solution was

$y = Ce^{t^3/3}$. The constant C may be determined by

inputting initial conditions and solving for C.

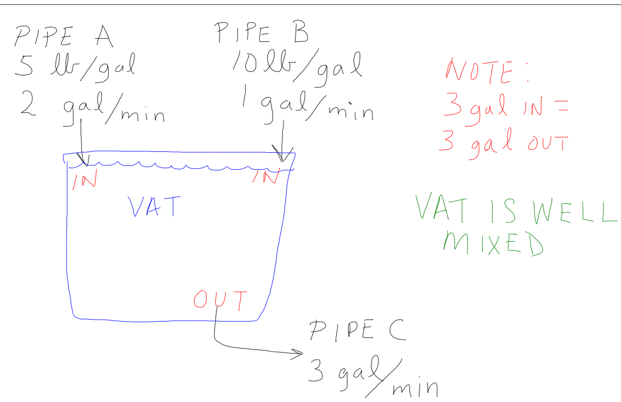
Mixture Problem

Suppose that in a large vat that has a capacity of 500 gallons, there are two inflow pipes and one outflow pipe and the total flow in is equal to the total flow out. Assume the vat is well mixed.

Suppose that 5 pounds of salt per gallon enter into pipe A at a rate of 2 gallons per minute and 10 pounds of salt per gallon enter into pipe B at a rate of 1 gallon per minute.

Salt water leaves the vat through pipe C at a rate of 3 gallons per minute.

We would like to obtain an equation that gives us S (salt) as a function of t (time).



To find an equation for S (salt) as a function of t(time), we first find an expression that gives a formula for dS/dt. We do this by realizing that rate of salt in or out per minute is given by

dS/st = Rate IN @ A +Rate IN @ B - Rate OUT @ C

THIS GIVES US

$$\frac{dS}{dt} = \underbrace{\frac{5\,lb}{gal} \times \frac{2\,gal}{min}} + \underbrace{\frac{10\,lb}{gal} \times \frac{1\,gal}{min}} - \underbrace{\frac{3\,gal}{min} \times \frac{S\,lbs}{500\,gal}}$$

$$\frac{dS}{dt} = \left(\frac{lb}{min}\ \text{IN}\right) + \left(\frac{lb}{min}\ \text{IN}\right) - \left(\frac{lb}{min}\ \text{OUT}\right)$$

NOTE: S = POUNDS OF SALT IN THE 500 gal. VAT.

THE SIMPLIFIED EQUATION IS

$$\frac{dS}{dt} = 20 - \frac{3S}{500}$$

TRY TO SOLVE BY SEPARATING AND INTEGRATING

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$$\int \frac{dS}{20 - .006S} = \int dt$$

$$\left(-\frac{1}{.006}\right) \text{LN}\,(20 - .006S) = t + C_1$$

$$\text{LN}\,(20 - .006S) = -.006t + C_2$$

WHERE $C_2 = -.006\,C_1$

$$e^{-.006t + C_2} = e^{-.006t} \cdot e^{C_2} = 20 - .006S$$

$$Ce^{-.006t} = 20 - .006S$$

WHERE $C = e^{C_2}$

$$S = \frac{20 - Ce^{-.006t}}{.006}$$

IF THERE ARE $S = 4$ POUNDS OF SALT INITIALLY, CAN YOU FIND THE VALUE OF C?

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LET $S=4$ & $t=0$

$$4=\frac{20-Ce^{-.006(0)}}{.006}$$

$$4=\frac{20-C}{.006}$$

$$\boxed{C=19.976}$$

THE SOLUTION FOR THESE CONDITIONS IS

$$S=\frac{20-19.976e^{-.006t}}{.006}$$

You may check your work by plotting this solution and the original differential equation on Graphmatica. Enter these as

dy = 20 - .006y {0,4}

and

y = (20-19.976*exp(-006x))/.006

From your graphs, what can you conclude about the amount of salt S in the tank as time approaches infinity? (BEWARE!)

Go back to your graph and repeatedly "zoom out" until the y-axis is in the 3000 to 3500 range.

Do you now see that the amount of salt in the tank approaches 20/.006 = 3333.333. . .as time approaches infinity. From the solution, isn't this what you would expect?

Do the exercises on P.30! Read the text!